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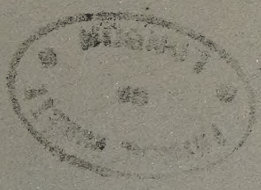

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Errata to Vol. VI. No. 2.

Page	56,	line	2	from	top,	for	iodoactic,	read	iodoacetic.
„	„,	„	10	„	„,	„	reactiou,	„	reaction.
„	57,	„	7	„	„,	„	lextro,	„	dextro.
„	„,	„	8	„	„,	„	daevo,	„	laevo.
„	61,	Table I.	$[\alpha]_o$	„	„,	„	−10.51,	„	−10.54.
„	„,	line	4	„	bottom,	„	decompound,	„	decomposed.
„	66,	„	2	„	top,	„	hydrochlyrie,	„	hydrochloric.
„	„,	„	6	„	„,	„	carlyl,	„	carbyl.
„	„,	„	7	„	„,	„	smine,	„	amine.
„	67,	„	4	„	bottom,	„	0.3208,	„	0.3280.
„	68,	„	2	„	„,	„	173–280°,	„	173–180°.
„	„,	„	18	„	„,	„	Pelonze,	„	Pelouze.
„	„,	„	21	„	„,	„	158–168°,	„	158–162°.
„	69,	„	2	„	„,	„	0.2068 and 0,8613,	„	2008 and 0.8618.
„	„,	„	6	„	„,	„	sodify,	„	solidify.

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Studies on Camphor Series II.

By

Shigeru Komatsu.

(Received June 7, 1922.)

In the previous article[1], Komatsu and Masumoto stated that the formation of lævo-isocamphan from dextro and lævo-camphors respectively would be explained by assuming an intermediate formation of camphene, and the present investigation was undertaken in co-operation with K. Fujii, S. Yamaguchi and R. Nakai to present some evidence in support of this assumption and also to solve such an interesting and extraordinary case in stereochemistry.

As a matter of fact, dextro-borneol was generally attended by lævo-isoborneol in the course of the reduction of dextro-camphor with metallic sodium in alcohol solution, and the reduction at high temperature is favorable for the formation of the iso-compound.[2] It is, therefore, natural to assume that the large quantity of dextro-camphor by the catalytic hydrogenation at 180° would be transformed into lævo-isoborneol of unstable character toward dehydrating agents, and consequently such treatment facilities the transformation of camphor into camphene.

In the transformation of dextro-camphor by the Grignard reaction into aryl derivatives of borneol, lævo-isobornyl compounds were only obtained as seen in J. Bredt's experiment.[3]

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1 First article; "Catalytic Reduction of Dextro-Camphor, I." Mem. Coll. Sci., Kyoto, **5**, 225 (1922).

2 E. Beckmann, J. prak. Chem., (2). **55**, 31 (1897).

3 J. prak. Chem., (2). **98**, 96 (1918).

Fujii, applying the Reformatzky synthesis on dextro-camphor with ethyl, lævo-menthyl and dextro-bornyl esters of iodoactic acid in absolute ether solution, using metallic magnesium as catalyser, has succeeded in isolating ethyl, menthyl and bornyl esters of bornyl acetic acid respectively. When these esters were saponified with barium hydroxide in alcohol solution, contrary to our expection, the resulting acid was the same and identical lævo-isobornyl acetic acid.

Dextro-camphor oxime, by reduction with metallic sodium in absolute amyl alcohol solution, yielded simultaneously dextro- and lævo-iso-bornyl amines in a proportion of 65·9 : 34·1 in the reactiou product as shown in Yamaguchi's experiment :

dextro-compound	dextro form	lævo-iso-form
$>C=O \longrightarrow >C<^{H}_{OH}$	81%	19%
$>C=NOH \longrightarrow >C<^{H}_{HN_2}$	66%	34%

Such fruitly yield of lævo-isocompound in the case of the oxime reduction compared with that of the ketone reduction under the same conditions, might be a proper way to regard the influence of the NH_2 group of more positive character than the OH group on the asymmetric transformation of the carbonyl carbon atom in camphor molecule.

In order to prepare only dextro-bornyl amine, avoiding the formation of the iso-compound, dextro-borneol was passed with dry ammonia gas on thoria heated at 400°, Nakai's efforts, however, did not bring the expected result; a trace of an amine compound could be detected by the carbyl amine reaction of the reaction product and the major portion of the borneol used was obserbed to be transformed into dextro-camphene. Lævo-borneol, in the same manner, was transformed into lævo-camphene.

To prove the camphene was derived from bornyl amine formed temporarily by the interaction of borneol and ammonia, but not directly from borneol by the Wagner rearrangement, dextro-bornyl amine was passed on the heated thoria and was observed by Yamaguchi to decompose entirely into dextro-camphene and ammonia.

The striking result here is the formation of lævo-isocamphan from both dextro-camphene and lævo-camphene by the catalytic hydrogenation with reduced nickel and hydrogen.

Thus, the formatian of lævo-isocamphan from both dextro-and

lævo-camphor by catalytic hydrogenation will be explained experimentally without any assumption as shown in the following scheme:

Dextro-camphor	reduction ⟶	dextro-borneol lævo-isoborneol	rearrangement ⟶
dextro-camphene	reduction ⟶	lævo-isocamphan	
Lævo-camphor	reduction ⟶	lextro-isoborneol dævo-borneol	rearrangement ⟶
lævo-camphene	reduction ⟶	lævo-isocamphan	

As to the existence of solid and liquid modifications in camphene and isocamphan the subject is now being studied in my laboratory and the result will be communicated in the next article.

I. SYNTHESIS OF LAEVO-ISOBORNYL ACETIC ACID AND ITS ESTERS.

BY

KITARO FUJII.

1. *Isobornyl Acetic Acid Ethylester,* $C_9H_{16}C<^{CH_2COOC_2H_5}_{OH}$.

It was prepared by the Reformatzky synthesis from 11 gm. metallic magnesium, 32 gm. dextro-camphor m.p. 177°, $[\alpha]_D = +44{\cdot}22°$ in alcohol solution, and iodo-acetic ester[1] of B.p. 83–85° at 2·3 mm. pressure obtained from chloro-acetic acid ethyl ester[2] B p. 144–145° $d_4^{25} = 1{\cdot}005$ and potassium iodide. The organo-magnesium compound thus formed, was decomposed with ice-cooled dilute sulphuric acid and the generated ester was extracted with ether. The ether solution, after distilling off the solvent, was subjected to steam distillation to remove the unchanged camphor and iodo-acetic ester. The residue in the flask was extracted again with ether, and the solution, after being dried with anhydrous sodium sulphate was subjected to fractional distillation under reduced pressure. The fraction, boiling at 108–112°. 3 mm was collected, was a light yellow viscous liquid. The

[1] J. Chem. Soc., **65**, 482 (1894).

[2] Lieb. Ann, **188**, 218 (1877).

yield was 10·2 gm. (20·2%). The specific rotatory power in 98% alcohol solution and the chemical composition of the ester were determined :

Substance	Solvent	α_D^{25} in 2 dm. tube	$[\alpha]_D^{25}$
0·2014 gm.	25 cc.	− 23.8′	− 24·62°
0·1890 ,,	,,	− 22·6′	− 24·91°
0·4231 ,,	,,	− 51·8′	− 25·51°
0.4330 ,,	,,	− 50·43′	− 24·82°
		mean	− 24·97°

Substance	CO_2	H_2O	C %	H %
0·1318	0·3471	0·1185	70·19	10·06
0·1058	0·2725	0·0937	70·24	9·91
0·0901	0·2292	0·0813	69·41	10·09
0·1672	0·4215	0·1476	68·77	9·90
		mean	69·65	9·99
		Calc. for $C_{14}H_{22}O_3$	69·95	10·06

2. *Laevo-Bornyl Acetic Acid Laevo-Menthyl Ester,*

$$C_9H_{16}C<\begin{matrix}CH_2COOC_{10}H_{18}\\OH\end{matrix}.$$

It was prepared by the interaction of 3·3 gm. dextro-camphor and 7·0 gm. iodo-acetic acid laevo-menthyl ester B. p. 110–130°, 4 mm. pressure, prepared from potassium iodide and chloro-acetic acid laevo-menthyl ester, in the presence of 1·2 gm. metallic magnesium, in 21 gm. absolute ether solution. The ester purified from menthol and iodo-acetic ester, was a syrupy substance, and was directly transformed by hydrolysis with barium hydroxide solution into free acid.

3. *Lævo-Isobornyl Acetic Acid Dextro-Bornylester,*

$$C_9H_{16}C<\begin{matrix}CH_2COOC_{10}H_{17}\\OH.\end{matrix}$$

Keeping the principle of an asymmetric synthesis in mind, an attempt to produce a dextro-bornyl derivative was undertaken by employing iodo-acetic acid dextro-bornyl ester instead of lævo-menthyl ester. Contrary to the expectation the resulting substance was a derivative of the same leavo-isobornyl acetic acid.

The ester was prepared by the interaction of 5 gm. metallic magnesium, 14·3 gm. dextro-camphor and 30 gm. iodoacetic acid dextro bornyl ester B.p. 130–140°, 3–5 m.m. pressure, $[\alpha]_D^{17} = +34{\cdot}6°$ in 98% alcohol solution, obtained from 30 gm. chloroacetic dextro bornylester[1] $[\alpha]_D^{17} = +30{\cdot}63°$ purified by distillation under reduced pressure and 20 gm. potassium iodide.

The free ester was purified by recrystallization from the dilute alcohol solution and found to be melted at 196–197°. The yield was 7·12 gm. (22·%).

Substance	CO_2	H_2O	C %	H %
0·1164 gm.	0·3258 gm.	0·1075 gm.	76·22	10·34
0·1031 „	0·2879 „	0·0997 „	76·03	10·57
		mean	76·13	10·46
		Calc. for $C_{22}H_{36}O_3$	75·80	10·45

Substance	Solvent alcohol	α_D^{18} 1 dm. tube	$[\alpha]_D^{18}$
0·1616 gm.	10 cc.	+ 16·6′	+ 17·12°
0·1322 „	10 „	+ 14·1′	+ 17·75°
		mean	+ 17·44°

1 C. R., **134**, 609 (1902).

4. *Laevo-Bornyl Acetic Acid*, $C_9H_{16}<\begin{matrix}CH_2COOH.\\OH.\end{matrix}$

1·5 gm. ethyl ester was saponified in 5 c.c. alcohol solution with 5 c.c. of 30% caustic potash solution, heating the mixture on a water bath for 4 hours, and the product was then treated with ether to remove the unchanged ester and other product. The aqueous solution acidified with ice-cooled dilute sulphuric acid, the generated free acid was extracted witn ether in the Sudō-Kumagawa apparatus.

The free acid was more conveniently prepared from the ethyl ester by saponifying with saturated barium hydroxide solution; an excess of barium hydroxide was removed by means of carbon dioxide from the solution and the filtrate from the barium carbonate, was evaporated to dryness on a water bath, The barium salt, thus formed was decomposed with 10% sulphuric acid, and the free acid was extracted with ether.

To the ether solution, dry ammonia gas was passed, the ammonium salt precipitated, was separated from the solution and was transformed into silver salt which was analysed on drying in a vacuum desiccator and the results are shown with those of barium salt in the following table.

Substance	CO_2	H_2O	Ag	C %	H %	Ag %
0·1212 gm.	0·1994 gm.	0·0687 gm.	0·0428 gm.	44·87	6·35	35·33
0·0963 ,,	——	——	0·0362 ,,	——	——	37·5
Calc. for $C_{12}H_{21}O_2Ag$				45·14	6·00	34·82

Substance	CO_2	H_2O	$BaSO_4$	C %	H %	Ba %
0·1303 gm.	0·1303 gm.	0·0931 gm.	——	54·17	6·66	——
0·2138 ,,	——	——	0·0804 gm.	——	——	22·14
Calc. for $(C_{21}H_{22}O_4)$ Ba				54·85	6·36	23 83

The barium salt of the same acid was also obtained from dextro-bornyl ester and laevo-menthyl ester respectively, by hydrolysing with

barium hydroxide in alcohol solution. The analytical results of the salt are shown in the following.

Sample	CO_2	H_2O	$BaSO_4$	C %	H_2 %	Ba %
0·1916 gm. (salt from *d*-bornyl ester)	——	——	0·0795 gm.	——	——	24·35
0·0569 ,, (salt from *l*-menthyl ester)	0·0536 gm.	0·0333 gm.	0·0238 ,,	55·71	6·55	24·62

The specific rotatory power of the barium salt in the aquous solution and of the free acid in 98% alcohol solution was determined:

I. Barium Salt.

Substance	Remark	Solvent	α_D^{18} 2dm. tube	$[\alpha]_D^{18}$
0·2077 gm.	from ethyl ester	25 c.c. H_2O	— 10·21′	— 12·91°
0·0998	from *d*-bornyl	,,	— 5′	— 10·51°
0·2138	ester	,,	— 12′	— 11·69°
0·1203	from *l*-menthyl ester	,,	— 5·50′	— 9·53°

II. Free Acid.

Substance	Remark	Solvent	α_D 1 dm. tube	$[\alpha]_D^{17}$
0·028 gm.	from ethyl ester	10 c.c. alcohol	— 2·4′	— 20·32°
0·2330 ,,	,, *d*-bornyl ester	25 c.c. ,,	— 27·6′	— 24·82°
0·1106 ,,		25 c.c. ,,	— 12·6′	— 25·62°
0·0399 ,,	,, *l*-menthyl ester	10 c.c. ,,	— 4′	— 19·85°

The free acid, soluble in water and alcohol, was observed to be decompound, when heated with conc. sulphuric acid into an unsaturated compound.

The barium and ammonium salts were soluble in water, but its silver salt insoluble in water.

II. BORNYLAMINE AND ITS TRAFSFORMATION INTO ISOCAMPHENE.

BY

SHOZO YAMAGUCHI.

For the preparation of bornylamine 16 gm. dextro camphor-oxime Mp. 118° were reduced following M. Forster's directions[1] with 20 gm. metallic sodium in 160 c.c. absolute isoamylalcohol solution. The reaction product acidified with hydrochloric acid, and isoamylalcohol was distilled off, whereas the amine hydrochloride crystallized out. By concentrating the mother liquor separated from the first crop, the amine salt was deposited. The total yield was 15 gm. The specific rotatory power of the two samples in absolute alcohol solution was determined, and the results were $[\alpha]_D^{17.5} = +16{\cdot}4$ and $[\alpha]_D^{17.5} = -5{\cdot}5$ respectively. The aqueous solution of an equimolecular mixture of bornylamine hydrochloride and silber dextro camphor sulphonate, filtered silver chloride off, was evaporated on a water bath to dryness and the resulting amine camphor sulphonate was recrystallized from anhydrous acetone solution to separate the dextro bornylamine salt from laevo-isobornylamine salt by the difference of their solubilities in the solvent. The purity of the isolated substance was determined by measuring the rotatory power of 10 c c. aqueous and absolute alcohol solutions. As seen in the following table, to attain desired purity of dextro-bornylamine dextrocamphor sulphonate, the crystallization should be repeated for seven times.

No. of time	Substance	Solvent	α_D^{18}	$[\alpha]_D^{18}$
5	0·1919gm.	water	+ 27′	+ 23·5°
6	0·2755 ,,	,,	+ 41′	+ 24·8°
7	0·1451 ,,	,,	+ 22·5′	+ 25·5°

[1] J. Chem. Soc., **73**, 386 (1898).

No. of time	Substance	Solvent	α_D^{18}	$[\alpha]_D^{18}$
7	0·2107 grm.	absolute alcohol	+ 47·8′	+ 37·5°
8	0·1914 grm.	„	+ 43·0′	+ 37·4°

For confirmation, the contents of the sulphur and nitrogen in the compound were determined:

0·1421 gm. substance gave 0·0866 gr. $BaSO_4$ 8·40% S
theory requires for $C_{20}H_{34}SNO_4$ 8·30% S
0·1010 gm. substance gave 4·00% N by the Kjeldahl
theory requires for $C_{20}H_{34}SNO_4$. 3·63% N.

The camphor-sulphonate with caustic potash solution, was hydrolysed, the generated free amine was extracted with ether, which was transformed into hydrochloride.

A solution of 0·1582 gm. substance in absolute alcohol and made up to 10 cc. $\alpha_D^{18} = +23·5'$ in 10 cm. tube; whence $[\alpha]_D^{18} = +24·7°$.

0·1246 gm. substance gave 0·094 gm. AgCl

	found	calc. for $C_{10}H_{20}NCl$
Cl	18·20	18·60

The authors' substance showed the higher specific rotatory power for 1·5° compared with Forster's, which might be considered to indicate the purity of the former.

To calculate the proportion of both dextro-bornylamine and laevo-isobornylamine formed simultaneously in the reducing product of the oxime, 1·5 gm. dextro-camphor-oxime was reduced as above mentioned, and 1·58 gm. hydrochloride of mixed amines were obtained which showed $\alpha_D = -2.$ in 1 dm. tube of its 50 c.c. absolute alcohol solution; whence $[\alpha]_D = -1·05°$.

Hence the percentage was:

d-bornylamine 65·9%
l-isobornylamine 34·1%

where *d*-bornylamine hydrochloride
$[\alpha]_D = +24·7°$ (Authers' data).
l-isobornylamine hydrochloride
$[\alpha]_D = -44·2°$ (Forsters' data).

12 gm. of the bornylamine hydrochloride of $[\alpha]_D = +16·4°$ dissolved in water, alkalined with caustic soda and then subjected to steam

distillation to get the amine in free state, and 6 gm. solid and 4 gm. liquid substance were obtained. Both solid and liquid forms separated mechanically, confirmed to be a mixture of bornylamine and isobornylamine in different proportions, by the determination of the rotatory power in absolute alcohol solution and of the nitrogen contents of their hydrochlorides:

Form	Substance	Solvent	Length of tube	α_D^{18}	$[\alpha]_D^{18}$
Liquid	0·2720 gm	10 c.c. abs. alcohol	1 dm.	+ 27′	+ 16·5°
Solid	0·2377 „	10 c.c. „	„	+ 28′	+ 19·6°

0·1005 gm. gave 7.81% N (by the Kjeldahl method).

Liquid { dextro-form 88%; laevo-iso-form 12%

0·1248 gm. gave 7·57% N (by the Kjeldahl method).

Solid { dextro form 92%; laevo iso form 8%

O. Wallach[1] has observed that when bornylamine heated at 200–210° with acetic acid anhydride was decomposed into camphene and ammonia. The question will naturally be developed from the idea of reversibility that such chemical change might be realized by the catalytic action of thoria or alumina on bornylamine, 7·5 gm. bornylamine m.p. 157·5–158° $[\alpha]_D^{18} = \frac{37{\cdot}5 \times 100}{0{\cdot}2998 \times 60 \times 10} = +42{\cdot}1°$, consisting of 96·2% dextro-bornylamine 3·8% laevoisobornylamine, were passed on thoria heated at 400°, and yielded 5·5 gm. liquid product.

Dextro-camphene was separated from the reaction product, after acidifying with hydrochloric acid, by steam distillation, and extracted with ether. The ether was distilled off from the solution, and the residue wss fractinated under ordinary pressure and collected, the fraction boiling at 160–167°. The yield was 2 gm. It was again distilled on metallic sodium and obtained the fraction boiling at 153–167° with a yield of 1 gm.

The specific rotatory power in absolute alcohol solution, density, index of refraction and chemical composition were determined.

[1] Lieb. Ann., **269**, 349 (1892).

Substance	a_D^{18} 1 dm. tube	$[a]_D^{18}$	d_4^{25}	n_4^2	M.R.
0·1949	+ 7′	5·9°	0·8591	1·4750	44·63

Substance	CO_2	H_2O	C %	H %
0·0986 gm.	0·3132 gm.	0·1039 gm.	86·63	11·83
0·1173 „	0·3531 „	0·1173 „	86·84	11·85
		mean	86·74	11·84
	Calc. for	$C_{10}H_{16}$	88·15	11·85

III. CATALYTIC PREPARATION OF CAMPHENE AND ISOCAMPHAN.

BY

RISABURO NAKAI.

An attempt to prepare bornylamine by the catalytic process proposed by Professor Paul Sabatier[1] from bornyl ethyl ether or bornyl acetate with ammonia was tried but to no purpose.[2]

To obtain the amine by the catalytic method from borneol and ammonia, 110 gr. dextro-borneol m.p. 203–204°, $[a]_D^{20} = \frac{19' \times 10}{0{\cdot}2093} = +15°{\cdot}13$ in alcohol solution (68·76% *d*-borneol 31·24% *l*-isoborneol[3]) prepared from dextro-camphor m.p. 177°, $[a]_D^{20} = \frac{51 \times '10}{0{\cdot}2033} = +41°{\cdot}81$, by reduction with metallic sodium, were passed with dry ammonia gas on thoria heated at 400°, and 103 gr. of the reaction product of

1 La Catalyse en Chimie Organique, 1920, p. 287.

2 Oppenheim und Precht, Ber. D. Chem. Ges., **9**, 325 (1876); N. Menschutkin, Ibid., **15**, 2512 (1882); J. Nef, J. Amer. Chem., **26**, 1549 (1904).

3 E. Beckmann, J. prak. Chem., (2), **55**, 31 (1897).

dark-yellow liquid were obtained. The product after being acidified with hydrochlyric acid, was subjected to steam distillation and the acidic residue was alkalined with caustic soda and distilled off a basic substance with steam. The basic distillate was extracted with ether and the ether solution having distilled off the ether, there remained a few milligrams of solid substance which gave the carlyl amine reaction, indicating the presence of a primary smine compound.

The first distillate, the main part of the reaction product, was distilled under ordinary pressure and collected the fraction 130–350° which amounted to 60 gr.

The fraction was fractionated under 764 mm. pressure 5 times and the following result were obtained.

Fraction	Yield	Remark
135–150°	2·5 gr.	colourless
150–158°	3·9 ,,	,,
158–162°	9·5 ,,	light-yellow
162–166°	4·8 ,,	,,
166–173°	7·3 ,,	,,
173–180°	3·5 ,,	yellow
180–205°	4·7 ,,	yellowish-brown
above 205°	13·7 ,,	dark-brown. containing borneol
	49·9 ,,	
loss	10·1 ,,	

The optical rotatory power of the alcohol solution, the density the index of refraction and the elementary analysis of each fraction were determined, and the results are shown in the following table:

Fraction	Substance	a_D^{20}, 1 dm. tube	$[a]_D^{20}$	$d_4^{25°}$	$n_D^{25°}$
135–150°			0	0·8736	1·4692
150–158°	0·2037 gr.	+ 11′·00	+ 9°·00	0·8825	1·4772
158–162°	0.1998 ,,	+ 12′·00	+ 10°·01	0·8630	1·4689

Fraction	Substance	α_D^{20}, 1 dm. tube	$[\alpha]_D^{20}$	$d_{4^\circ}^{25^\circ}$	$n_D^{25^\circ}$
162–166°	0·1957 gr.	+ 10′·00	+ 8°·52	0·8653	1·8080
166–173°	0·2064 „	+ 5′·25	+ 4°·24	0·8869	1·4747
173–180°	0·2151 „	+ 10′·25	+ 7°·94	0·8951	1·4797
180–205°	0·2075 „	+ 22′·00	+ 17°·67	0·9016	1·4807

Fraction	Substance	CO_2	H_2O	C %	H %
135–150°	0·1113 gr.	0·3432 gr.	0·1151 gr.	84·10	11·57
	0·0932 „	0·2876 „	0·0956 „	84·16	11·48
	0·1012 „	0·3113 „	0·1031 „	83·89	11·40
		mean		84·05	11·48
150–158°	0·1145 „	0·3561 „	0·1186 „	84·86	11·60
	0·1030 „	0·3202 „	0·1086 „	84·78	11·80
	0·1172 „	0·3663 „	6·1208 „	85·24	11·53
		mean		84·96	11·64
158–162°	0·1082 „	0·3463 „	0·1139 „	87·33	11·79
	0·1134 „	0·3654 „	0·1204 „	87·88	11·88
	0·1266 „	0·4059 „	0·1346 „	87·43	11·90
		mean		87·55	11·85
162–166°	0·1070 „	0·3389 „	0·1123 „	86·38	11·74
	0·1196 „	0·3810 „	0·1232 „	86·88	11·53
	0·0895 „	0·2844 „	0·0924 „	86·67	11·55
		mean		86·64	11·61
166–173°	0·1069 „	0·3208 „	0·1085 „	83·68	11·36
	0·1107 „	0·3395 „	0·1109 „	83·63	11·21
	0·1068 „	0·3284 „	0·1081 „	83·86	11·33
		mean		83·72	11·30

Fraction	Substance	CO_2	H_2O	C %	H %
173–280°	0·1054 gr.	0·3177 gr.	0·1043 gr.	82·25	11·08
	0·1149 ,,	0·3477 ,,	0·1123 ,,	82·53	10·94
	0·1131 ,,	0·3426 ,,	0·1103 ,,	82·65	10·92
		mean		82·48	10·98
180–205°	0·1042 ,,	0·3094 ,,	0·1012 ,,	81·02	10·87
	0·1077 ,,	0·3213 ,,	0·1037 ,,	81·36	10·78
	0·1143 ,,	0·3403 ,,	011·04 ,,	81·20	10·81
		mean		81·19	10·82

As seen in the above tables, the first fraction B. p. 135–150°, consists of an aromatic hydrocarbon and the last fraction B.p. 180–205°, a mixture of borneol and a small quantity of hydrocarbon $C_{10}H_{16}$, which showed a higher specific rotatory power than that of the borneol used indicating that the lævo-isoborneol associated with dextro-borneol, was partially removed by the catalytic action of thoria and transforming itself into comphene. The third fraction B.p. 158–162°, consisting of pure comphene, requires C = 88·15 % ; H = 11·85 %, which was formerly obtained by M. Pelonze,[1] G. Kachler,[2] O. Wallach[3] and N. Zelinsky & T. Zelikow[2] from borneol by the action of various dehydrating agents.

The liquid dextro-camphene, B.p. 158–162°, thus obtained was distilled on metallic sodium, and the fraction B.p. 158–168°, the colourless liquid, was observed to possess the chemical composition and physical properties as shown in the following table :—

Substance	CO_2	H_2O	C %	H %
0·1080 gr.	0·3479 gr.	0·1139 gr.	87·86	11·80
0·1042 ,,	0·3366 ,,	0·1094 ,,	88·10	11·75
	mean		87·98	11·78

[1] Lieb. Ann., **40**, 326 (1841); C.R., **11**, 365 (1840).
[2] Lieb. Ann., **164**, 75 (1872).
[3] Lieb. Ann., **230**, 225 (1885).
[4] Ber. D. Chem. Ges., **34**, 3249 (1901).

Substance	α_D^{20}, 1 dm. tube	$[\alpha]_D^{20}$	$d_{4^\circ}^{25^\circ}$	$n_D^{25^\circ}$	M.R.
0·2068 gr.	+ 12′	+ 9°·96	0·8613	1·4689	43·98

In another experiment, two fractions (I) B.p. 155–157° and (II) B.p. 158–160° which were considered to consist of liquid camphene, on standing from two weeks at ordinary temperature, after being treated with metallic sodium, were observed to sodify. The crystals obtained from the first fraction, melted at 50° and showed the specific rotatory power $[\alpha]_D^{20} = +17^\circ{\cdot}61$ in ethyl alcohol solution.

The author has obtained lævo-camphene on applying the same treatment to lævo borneol as to the dextro compound; using 43 gr. lævo-borneol, m.p. 202·5–203°, $[\alpha]_D^{20} = \frac{45' \times 10}{0{\cdot}2027} = -37^\circ{\cdot}01$, 41 gr. of the reaction product were obtained, and this was purified by fractional distillation 4 times under 757–758 mm. pressure:

Fraction	Yield	Remark
Below 150°	1·0 gr.	light yellow
150–157°	4·9 ,,	,,
157–163°	9·4 ,,	yellow
163–166°	2·5 ,,	,,
166–173°	1·5 ,,	,,
	19·3 ,,	

Each fraction was redistilled with metallic sodium, and the following results were obtained:

Fraction	Yield	Remark
140–154°	2·1 gr.	colourless
154–158°	5·5 ,,	,,
158–163°	7·4 ,,	,,
163–170°	2·6 ,,	,,
above 170°	0·7 ,,	yellow
	18·3 ,,	
loss	1·0 ,,	

The analytical results and physical constants of the two fractions, (I) B.p. 154–158°; (II) B.p. 158–163°, are shown in the following table:

Fraction	Substance	CO_2	H_2O	C %	H %
154–158°	0·1045 gr.	0·3323 gr.	0·1087 gr.	86·72	11·64
158–163°	0·1054 „	0·3392 „	0·1097 „	87·77	11·65

Fraction	Substance	α_D^{20}, 1 dm. tube	$[\alpha]_D^{20}$	$d_{4^\circ}^{25^\circ}$	$n_D^{25^\circ}$	M.R.
154–158°	0·2040 gr.	− 8′·0	− 6°·54	0·8645	1·4674	43·73
158–163°	0·20475 „	− 4′·5	− 3°·66	0·8619	1·4684	43·94

Camphene	B.p.	$[\alpha]_D^{20}$	$d_{4^\circ}^{25^\circ}$	$n_D^{25^\circ}$	M.R.
from *d*-borneol	158–168°	+ 9·°96	0·8618	1·4689	43·98
l-borneol	158–163°	− 3·°66	0·8619	1·4684	43·94
d-bornylamine (by S. Yamaguchi)	150–167°	+ 5·°90	0·8591	1·4750	44·63

The fraction B.p. 158–163° was observed to consist of pure camphene found in the corresponding fraction obtained from dextro-borneol. The fact that camphene was derived from bornylamine as observed by S·. Yamaguchi, has induced the author to consider that the formation of camphene from borneol by the catalytic action of thoria in the atomosphere of ammonia gas would be explained easily, admitting the intermediate formation of bornyl amine.

Professor P. Sabatier and Senderens[1] have obtained liquid hydrocarbon $C_{10}H_{18}$ B.p. 164–165°, $d_{4^\circ}^{0^\circ}$=0·849 by the catalytic reduction of camphene, N. Zelinsky[2] and Nametkine and Abakoumovskaia,[3] however,

[1] C.R., **132**, 1254 (1901).

[2] F. W. Semmler: Die Ätherischen Öle, II., p. 70 (1906).

[3] P. Sabatier: La Catalyse en Chimie Organique, p. 182, (1920).

have reported that solid camphan was obtained from camphene by the same process.

The author has transformed camphene following Professor P. Sabatier's directions into liquid isocamphan; 3 gr. dextro camphene B.p. 158–168° purified by distillation on metallic sodium, was reduced with nickel and hydrogen, and 1·5 gr. of reaction product were obtained which was purified by distillation, and of the fraction B.p. 164–167° the optical rotatory power, the density, the index of refraction and the chemical composition were determined.

Substance	CO_2	H_2O	C %	H %
0·0943 gr.	0·2994 gr.	0·1154 gr.	86·59	13·69
0·1013 „	0·3220 „	0·1227 „	86·69	13·55
		mean	86·64	13·62
		Calc. for $C_{10}H_{18}$	86·87	13·13

Substance	a_D^{20}, 1 dm. tube	$[\alpha]_D^{20}$	$d_{4^\circ}^{25^\circ}$	$n_D^{25^\circ}$	M.R.
0·20795 gr.	− 4′·5	− 3°·61	0·8386	1·4534	44·56

Both fractions (I) B.p. 154–158°; (II) B.p. 158–163° of laevo-camphene, obtained from lævo-borneol, were reduced by the catalytic process, and 1 gr. the purified substance B.p. 160–164°, 750 mm. from 2 gr. fraction (I) and 1 gr. B.p. 160–163°, 1·5 gr. isocamphan B.p. 163–165°, 750 mm. from 3 gr. fraction (II) were isolated. Of each purified substance, the physical properties and the chemical composition were determined:

Fraction	Substance	CO_2	H_2O	C %	H %
160–164° (from I.)	0·10505 gr.	0·33415 gr.	0·1270 gr.	86·77	13·53
163–165° (from II.)	0·1096 „	0·34775 „	0·1317 „	86·53	13·45
		Calc. for $C_{10}H_{18}$		86·87	13·13

Fraction	Substance	α_D^{20}, 1 dm. tube	$[\alpha]_D^{25°}$	$d_{4°}^{25°}$	$n_D^{25°}$	M.R.
160–164° (from I.)		0	0	0·8460	1·4451	43·47
163–165° (from II.)	0·2039 gr.	− 4′	−3°·27	0·8440	1·4492	43·92

The interesting results here obtained are that the same lævo isocamphan was derived from dextro- and lævo-camphenes respectively by catalytic hydrogenation.

May, 1922, Laboratory of Organic Chemistry.

Reaction between Aniline and Chloracetic acid, I.

By

Ryuzaburo Nodzu and Shigeru Komatsu.

(Received June 7, 1922.)

When 20 c.c. aniline (1·04 mol.) and 20 grm. chloracetic acid (1 mol.) were mixed in a pressure bottle, white sublimate of chloracetanilide was observed to form immediately.[1] The mixture was heated in a boiling water bath for one hour, and the reaction product treated with 250 c.c. water and was left to stand over night at ordinary temperature, and then filtered.

The residue (I) suspended in 50 c.c. water, was digested with conc. ammonium carbonate solution on a water bath, cooled and filtered, the extraction being repeated once more to separate completely the acidic substance from the neutral reaction products.

The combined filtrate, on acidifying with dilute hydrochloric acid, yielded a white precipitate that amounted to 1·5 grm. The substance having been purified, by reprecipitation with hydrochloric acid from its ammonium salt solution, and by recrystallization from the 80 % alcohol solution, melted at 212° and gave on analysis the following results:

I. 0·1449 grm. substance gave 0·3561 grm. CO_2 and 0·0770 grm. H_2O.
II. 0·1498 grm. ,, ,, 12·4 c.c. N_2 at 15° and 764·6 m.m.

	Carbon	Hydrogen	Nitrogen
Calc. for $C_{16}H_{16}N_2O_3$	67·57	5·68	9·86
Found	67·02	5·95	9·78

[1] P. Meyer, Ber. D. Chem. Ges., 8, 1152 (1875).

The physical and chemical properties of the substance so far as we studied agree with those of phenylimidodiacetic acid mono-anilide which was obtained together with phenylimidodiacetanilide by A. Hausdörfer[1] from the reaction product of phenylimidodiacetic acid and aniline at 170°.

The authors, however, expected that the substance was formed by the interaction of phenylglycine and chloracetanilide. For the purpose of testing this view, an equimolecular mixture of the two substances was heated in a sealed tube in a boiling water bath for one hour and phenylimidodiacetic acid mono anilide was isolated and thus our assumption has been realized.

From the residue (II) insoluble in ammonium carbonate solution, a small quantity of diphenyldiketopiperazine and 1 gm. chloracetanilide were isolated, by extraction of the latter in the cold with 94 % alcohol leaving the former compound in the residue. Chloracetanilide thus obtained from the alcoholic solution, was purified by recrystallisation from hot aqueous solution, melted at 134·° and gave on analysis the following results:

I. 0·1005 grm. substance gave 0·2108 grm. CO_2 and 0·0479 grm. H_2O.
II. 0·1523 grm. „ „ 0·1251 grm. AgCl.

	Carbon	Hydrogen	Chlorine
Calc. for C_8H_8NOCl	56·63	4·75	20·92
Found	57·22	5·20	20·30

The pure diphenyldiketopiperazine, obtained from the insoluble residue in 94% alcohol and purified by the recrystallization from hot alcoholic solution, melted at 263° and gave on analysis the following results:

I. 0·1462 grm. substance gave 0·3844 grm. CO_2 and 0·0708 grm. H_2O.
II. 0·1462 grm. „ „ 13·4 c.c. N_2 at 15° and 763·2 m.m.

	Carbon	Hydrogen	Nitrogen
Calc. for $C_{16}H_{14}N_2O_3$	72·14	5·30	10·52
Found	71·71	5·44	10·81

The filtrate separated from the residue (I), indicates a strong acidic reaction, was neutralized with ammonia and concentrated under reduced pressure to its half volume. The concentrated solution being alkalined with ammonia the unchanged aniline was extracted with 30 c.c. ether and then acidified with dilute hydrochloric acid. The acid solution was

[1] Ber. D. Chem. Ges., **22**, 1795 (1889); **23**, 1990 (1890).

treated with 30 c.c. ether, the operation being repeated twice to expel completely phenylimidodiacetic acid, and again neutralized with ammonia water, and concentrated to 50 c.c. under reduced pressure. On standing the concentratnd solution, after it was acidified with dilute hydrochloric acid, for one day at ordinary temperature, phenylglycine crystallized out. The yield was 3 grm. It melted at 126–127°, and gave the following analytical results:—

I. 0·2398 grm. substance gave 0·5592 grm. CO_2 and 0·1310 gr. H_2O.
II. 0·1560 grm. ,, ,, 13·55 c.c. N_2 at 15° and 757·8 mm.

	Carbon	Hydrogen	Nitrogen
Calc. for $C_8H_9NO_2$	63·55	6·00	9·27
Found	63·60	6·11	9·73

Distilling off the ether, from the ether extract, there remained 3 grm. phenylimidodiacetic acid[1] which purified by crystallisation from the water solution, melted at 155° and gave on analysis the following results:

I. 0·1838 grm. substance gave 0·3870 grm. CO_2 and 0·0890 grm. H_2O.
II. 0·1686 grm. ,, ,, 9·9 c.c. N_2 at 14° and 762·4 m.m.

	Carbon	Hydrogen	Nitrogen
Calc. for $C_{10}H_{11}NO_4$	57·39	5·30	6·71
Found	57·42	5·41	6·94

Of the reaction products between aniline and chloracetic acid at 100°, we have actually isolated chloracetanilide, diphenyldiketopiperazine, phenylimidodiacetic acid mono-anilide, phenylglycine and phenylimidodiacetic acid.

The chemical reactions involving the synthesis of these substances will be classified in (*a*) the dehydration and (*b*) the dehydrochlorination as formulated below:

$$C_6H_5NH_2 + ClCH_2COOH \begin{cases} \nearrow ClCH_2COOH.NH_2C_6H_5 = ClCH_2CONHC_6H_5 + H_2O. \quad \ldots\ldots\ldots(a) \\ \searrow C_6H_5NH_2ClCH_2COOH = C_6H_5NH_2CH_2COOH + HCl. \quad \ldots\ldots(b) \end{cases}$$

Hydrochloric acid accelerates the reaction[2] (*a*) and on the other hand aniline may produce a similar effect on the reaction (*b*). Even in the absence of any catalyst, when aniline comes into contact with

1 P. Meyer, Ber. D. Chem. Ges., **14**, 1325 (1881); Hausdörfer, Loc. Cit.
2 H. Goldschmidt and C. Wachs, Zs. Physik. Chem., **24**, 364 (1897).

chloracetic acid in a sealed tube at 100°, and the reaction is started, one stimulates the other, they will proceed to their equilibrium states. The first reaction which takes place instantaneously after the reacting components are mixed would be easily decided to be (*a*) by the readiness of formation of the molecular compound. The presence of aniline which is capable forming an addition compound with hydrochloric acid may naturally accelerate the formation of phenylglycine from aniline and chloracetic acid, of phenylimidodiacetic acid from phenylglycine and chloracetic acid. Consequently, chloracetanilide formed as an intermediated product, in the presence of an excess of aniline, without remaining in the reaction products would be entirely transformed into phenylglycineanilide &c. as will seen in A. Hausdörfer's article. Furthermore, aniline plays the same part in the formation of diphenyldiketopiperazine from chloracetanilide and of phenylimidodiacetic acid mono-anilide from chloracetanilide and phenylglycine, as caustic potash does in the formation of the former as observed by Abenius,[1] and also by Prof. Kuhara and Chikashige.[2]

April, 1922, Laboratory of Organic and Biochemistry.

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[1] Ber. D. Chem. Ges., **21**, 1665 (1888).

[2] Mem. Coll. Sci. Engin. Kyoto, 1, 1 (1903–8).

On the Projective Deformation of Curved Surfaces. III.

By

Jôyô Kanitani.

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§ 1.

In a preceding paper[1] we proved that if a congruence I' conjugate to a surface (S) is a W-congruence and the asymptotic lines of the focal surfaces of that congruence and the surface (S) correspond, the surface (S) and the focal surfaces admit the projective deformation and, moreover, the conjugate net on (S) which corresponds to the developable surfaces of the congruence is that of projective deformation.

In this paragraph, we shall prove the same for congruence I'' which is conjugate to a surface (S), that is, for a congruence which has the following relations with (S).

1° The lines of the congruence are in a one to one correspondence to the points of (S), accordingly to a surface of the congruence correspond a curve on (S).

2° A line of the congruence passes through the corresponding point on (S).

3° To the developable surfaces of the congruence correspond a conjugate net on (S), in other words, the developable surfaces of the congruence cut (S) in a conjugate net.

Let us take as parameter curves the conjugate net which corresponds to the developable surfaces of the congruence and suppose that (S) is given by a completely integrable system of partial differential equations

[1] J. Kanitani, Mem. Coll. Sci., Kyoto, **6**, 129 (1922).

$$\left.\begin{aligned} \frac{\partial^2 x}{\partial u^2} &= a\frac{\partial^2 x}{\partial v^2} + b\frac{\partial x}{\partial v} + c\frac{\partial x}{\partial v} + dx, \\ \frac{\partial^2 x}{\partial u \partial v} &= b'\frac{\partial x}{\partial u} + c'\frac{\partial x}{\partial v} + d'x, \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (1)$$

of which one of the conditions of complete integrability is

$$\frac{\partial}{\partial v}(b + 2c') = \frac{\partial}{\partial u}\left(\frac{2ab' - c - a_v}{a}\right).$$

Then the congruences whose developables cut (S) in the curves C_u (v-const.), C_v (u-const.) are those which are formed by the lines which connect a point (x) and a point (z) such that[1]

$$z = \theta\frac{\partial^2 x}{\partial v^2} - \frac{1}{a}\left(\frac{\partial \theta}{\partial u} + c'\theta\right)\frac{\partial x}{\partial u} - \left(\frac{\partial \theta}{\partial v} + \frac{ab' - c - a_v}{a}\theta\right)\frac{\partial x}{\partial v},$$

where θ is any one of the solutions of the partial differential equation

$$\frac{\partial^2 \theta}{\partial u \partial v} = -\left(b' - \frac{a_v}{a}\right)\frac{\partial \theta}{\partial u} - c'\frac{\partial \theta}{\partial v} + \left(d' + \frac{\partial b'}{\partial u} - \frac{\partial c'}{\partial v} - \frac{\partial b}{\partial v} + \frac{c'}{a}\frac{\partial a}{\partial v}\right)\theta. \quad \cdots\cdots\cdots\cdots (2)$$

Focal surfaces are those which are described by the points ω and ν such that

$$\omega = z - \xi x,$$
$$\nu = z - \xi' x.$$

where

$$\xi = -\frac{1}{a}\frac{\partial^2 \theta}{\partial u^2} - \frac{1}{a}\left(b + c' - \frac{1}{a}\frac{\partial a}{\partial u}\right)\frac{\partial \theta}{\partial u} - b'\frac{\partial \theta}{\partial v} + \left[\frac{\partial b'}{\partial v} - \frac{\partial}{\partial u}\left(\frac{c'}{a}\right) - \frac{1}{a}(bc' - b'c) + \frac{b'}{a}\frac{\partial a}{\partial v}\right]\theta$$

$$\xi' = -\frac{\partial^2 \theta}{\partial v^2} - \frac{c'}{a}\frac{\partial \theta}{\partial u} - \frac{ab' - c - a_v}{a}\frac{\partial \theta}{\partial v} + \left[\frac{1}{a}\frac{\partial c'}{\partial u} - \frac{\partial b'}{\partial v} - \frac{1}{a}(bc' - b'c) - \frac{ca_v}{a^2} + \frac{\partial^2}{\partial v^2}\log a - d\right]\theta.$$

From the above formulas we have

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[1] J. Kanitani, Mem. Coll. Sci., **5**, 329 (1922).

$$\left.\begin{aligned}\frac{\partial\omega}{\partial u}&=-\left(\frac{\partial\zeta}{\partial u}+c'\zeta\right)x\\ \frac{\partial\omega}{\partial v}&=-\zeta\frac{\partial x}{\partial v}+b'\zeta x.\end{aligned}\right\}\quad\ldots\ldots\ldots

where

$$A=\frac{a\left(\frac{\theta_v}{\theta}+b'+\frac{a_v}{a}\right)}{\left(\frac{\zeta_v}{\zeta}+b'\right)}.$$

Similarly, we have an equation of the form

$$\frac{\partial^2 w}{\partial u^2}=A_1\frac{\partial^2 w}{\partial v^2}+B_1\frac{\partial^2 w}{\partial u}+C_1\frac{\partial w}{\partial v}+D_1 w,$$

where

$$A_1=\frac{a\left(c'+\frac{\zeta_u}{\zeta}\right)}{\left(\frac{\theta_u}{\theta}+c'\right)}.$$

If the congruence in question is a W-congruence and the asymptotic lines of the focal surfaces of that congruence and (S) correspond, we must have

$$a=A=A_1,$$

which is equivalent to

$$\left.\begin{aligned}\frac{\zeta_u}{\zeta}&=\frac{\theta_u}{\theta},\\ \frac{\zeta_v}{\zeta}&=\frac{\theta_v}{\theta}-\frac{a_v}{a}.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

From (6) we have

$$\frac{\partial^2}{\partial u\partial v}\log a=0,$$

or

$$a=\frac{U(u)}{V(v)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

where $U(u)$ and $V(v)$ are functions of u only and v only respectively.

By the transformation

$$\bar{u}=\int\sqrt{U(u)}\,du,$$

$$\bar{v}=\int\sqrt{-V(v)}\,dv,$$

we can reduce a to -1.

Let us suppose that we have performed the above transformation. Then we have

$$\zeta = k\theta \qquad (k\text{: const.})$$

$$\xi = \frac{\partial^2\theta}{\partial u^2} + (b + c')\frac{\partial\theta}{\partial v} - b'\frac{\partial\theta}{\partial v} + \left[b'_v + c'_u + (bc' - b'c)\right]\theta,$$

$$\xi' = -\frac{\partial^2\theta}{\partial v^2} + c'\frac{\partial\theta}{\partial v} - (b' + c)\frac{\partial\theta}{\partial v} + \left[bc' - bc' - (b'_v + c'_u) + d\right]\theta,$$

and from (2) and (5)

$$\frac{\partial b}{\partial v} = 2\frac{\partial b'}{\partial u}. \qquad \text{(8)}$$

Accordingly, θ must satisfy the system of the equations.

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = +b\frac{\partial\theta}{\partial u} - c\frac{\partial\theta}{\partial v} - \left[d + 2(b'_v + c'_u)\right]\theta,$$

$$\frac{\partial^2\theta}{\partial u\partial v} = -b'\frac{\partial\theta}{\partial v} - c\frac{\partial\theta}{\partial v} + \left[d - b'_u - c'_v\right]\theta.$$

This system is completely integrable in virtue of (7), (8) and the conditions of the complete integrability of (1).

From (7) and (8) we know that the surface (S) admits the projective transformation and the curves C_u and C_v form the conjugate net of the projective deformation.

The focal surface which is described by ν is given by the system

$$\frac{\partial^2\nu}{\partial u^2} + \frac{\partial^2\nu}{\partial v^2} = B\frac{\partial\nu}{\partial u} + C\frac{\partial\nu}{\partial v} + D\nu,$$

$$\frac{\partial^2\nu}{\partial u\partial v} = \left(b' + \frac{\theta_u}{\theta}\right)\frac{\partial\nu}{\partial u} + \left[c' + \frac{1}{b'\theta + \theta_v}\frac{\partial}{\partial u}(b'\theta + \theta_v)\right]\frac{\partial\nu}{\partial v},$$

where

$$B = b + \frac{2\theta_u}{\theta}.$$

Therefore we know that this surface also admits the projective deformation.

Similarly, we can see that the other focal surface also admits the projective deformation.

§ 2.

We have, hitherto, supposed that a surface is given and considered congruence I and I' conjugate to that surface.

We shall, hereafter, suppose that a congruence is given and consider surfaces which have the following relations with that congruence.

1° There are one to one correspondences between the lines of given congruence and points on a surface (S).

2° A tangent plane at a point on (S) passes through the corresponding line of the congruence.

3° To the developable surfaces of the congruence correspond a conjugate net on (S). (8)

Wilcznski[1] has showed that the general theory of the congruence may be based upon the theory of the following completely integrable system of the partial differential equations.

$$\left.\begin{aligned} &\frac{\partial y}{\partial v} = mz, \qquad \frac{\partial z}{\partial u} = ny. \\ &\frac{\partial^2 y}{\partial u^2} = ay + bz + c\frac{\partial y}{\partial u} + d\frac{\partial z}{\partial v}, \\ &\frac{\partial^2 z}{\partial v^2} = a'y + b'z + c'\frac{\partial y}{\partial u} + d'\frac{\partial z}{\partial v}. \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (9)$$

The system (8) has precisely four pairs of linearly independent solutions (y_k, z_k), $(k=1, 2, 3, 4)$, such that the general solution will be of the form

$$y = \sum_{k=1}^{4} C_k y_k \qquad z = \sum_{k=1}^{4} C_k y_k$$

Let $y_1, \ldots\ldots, y_4$ and $z_1, \ldots\ldots, z_4$ be interpreted as the homogeneous coordinates of two points P_y and P_z.

As u and v vary, these points will describle two surfaces S_y and S_z respectively, and the line P_yP_z will generate a congruence whose focal surfaces consist of the two surfaces S_y and S_z.

Let us assume S_y and S_z are not degenerate.

By the transformations

$$\bar{u} = \phi(u), \qquad \bar{v} = \psi(v),$$

[1] Trans. Amer. Math. Soc., **16**, 310–327 (1915).

which do not alter the congruence, system (9) is transformed into a system of the same form. Denoting its coefficients by $\bar{m}$, $\bar{n}$, $\bar{a}$, etc., we have

$$\bar{m}=\frac{m}{\phi_v}, \qquad \bar{n}=\frac{n}{\phi_u},$$

$$\bar{d}=\frac{\phi_v}{\phi_u^2}d, \qquad \bar{c}'=\frac{\phi_u}{\phi_v^2}c'.$$

The condition of the complete integrability of the system (9) are

$$\left.\begin{aligned} &c=f_u, \qquad d'=f_v, \qquad b=-d_v-df_v,\\ &a'=-c'_u-c'f_u, \qquad mn-c'd=f_{uv},\\ &m_{uu}+d_{vv}+df_{vv}+d_vf_v-f_um_u=ma+db',\\ &n_{vv}+c'_{uu}+c'f_{uu}+c'_uf_u-f_vn_v=c'a+nb',\\ &2m_un+mn_u=a_v+f_umn+a'd,\\ &m_vn+2mn_v=b'_u+f_vmn+bc'. \end{aligned}\right\} \quad (10)$$

By the relation 3° of (8), the curves C_u (v const.) C_v (u-const.) on (S) form a conjugate net and the given congruence is a congruence Γ of (S) with respect to that conjugate net. Therefore, to the line which passes through points (y) and (z), corresponds a point ($\bar{x}$) on (S), and tangents at ($\bar{x}$) to the curves C_u and C_v which meet at ($\bar{x}$), pass through the focal points of this line, *i.e.*, the points (y) and (z) respectively.[1]

Hence ($\bar{x}$) is determined by the equations

$$\frac{\partial\bar{x}}{\partial u}=\lambda\bar{x}+\mu y, \qquad \text{..... (11)}$$

$$\frac{\partial\bar{x}}{\partial v}=\lambda_1\bar{x}+\mu_1 z. \qquad \text{..... (12)}$$

Differentiating (11) by v and (12) by u, we have

$$\frac{\partial}{\partial v}\left(\frac{\partial\bar{x}}{\partial u}\right)=\left(\lambda\lambda_1+\frac{\partial\lambda}{\partial v}\right)\bar{x}+(\lambda\mu_1+\mu m)z+\frac{\partial\mu}{\partial v}y,$$

$$\frac{\partial}{\partial u}\left(\frac{\partial\bar{x}}{\partial v}\right)=\left(\lambda\lambda^1+\frac{\partial\lambda_1}{\partial u}\right)\bar{x}+(\lambda_1\mu+\mu n)y+\frac{\partial\mu_1}{\partial u}z.$$

[1] J. Kanitani, Mem. Coll. Sci., **5**, 329 (1922).

Equating these two values of $\frac{\partial^2 \bar{x}}{\partial u \partial v}$, we have

$$\frac{\partial \lambda}{\partial v} = \frac{\partial \lambda_1}{\partial u},$$

$$\frac{\partial \mu}{\partial v} = \lambda_1 \mu + \mu_1 n,$$

$$\frac{\partial \mu_1}{\partial n} = \lambda \mu_1 + \mu m.$$

From these equations, we see that there is a function θ such that

$$\lambda = -\frac{\theta_u}{\theta}, \qquad \lambda_1 = -\frac{\theta_v}{\theta}.$$

$$\theta \frac{\partial \mu}{\partial v} + \mu \frac{\partial \theta}{\partial v} = \theta \mu_1 n,$$

$$\theta \frac{\partial \mu_1}{\partial u} + \mu_1 \frac{\partial \theta}{\partial v} = \theta \mu m.$$

Substituting the above values λ, λ_1 in (11) and (12), we know that $\bar{x}$ is given by the equations

$$\theta \frac{\partial \bar{x}}{\partial u} + \frac{\partial \theta}{\partial u} \bar{x} = \theta \mu y,$$

$$\theta \frac{\partial \bar{x}}{\partial v} + \frac{\partial \theta}{\partial v} \bar{x} = \theta \mu_1 z.$$

Put

$$x = \theta \bar{x}, \qquad p = \mu \theta, \qquad q = \mu_1 \theta.$$

Then we know that x is given by the equations

$$\left.\begin{aligned} \frac{\partial x}{\partial u} &= py, \\ \frac{\partial x}{\partial v} &= qz. \end{aligned}\right\} \quad \cdots\cdots (13)$$

where the functions p, q are a pair of solutions of the system of the partial differential equations

$$\left.\begin{aligned} \frac{\partial p}{\partial v} &= nq, \\ \frac{\partial q}{\partial n} &= mp. \end{aligned}\right\} \quad \cdots\cdots (14)$$

As by assumption, both S_y and S_z are not degenerate, the four points (y), (z), $\left(\frac{\partial y}{\partial u}\right)$, $\left(\frac{\partial z}{\partial v}\right)$ are not coplanar.

Therefore, x may be expressed in the form

$$x = w_1 y + w_2 z + w_3 \frac{\partial y}{\partial u} + w_4 \frac{\partial z}{\partial v}.$$

By (9) and (13) the above functions w_1, w_2, w_3, w_4 must satisfy the system of the equations

$$\left.\begin{aligned}
\frac{\partial w_1}{\partial u} + n w_2 + a w_3 + n_v w_4 &= p,\\
\frac{\partial w_2}{\partial u} + b w_3 + m n w_4 &= 0,\\
\frac{\partial w_3}{\partial u} + c w_3 + w_1 &= 0,\\
\frac{\partial w_4}{\partial u} + d w_3 &= 0,\\
\frac{\partial w_1}{\partial v} + w_3 m n + a' w_4 &= 0,\\
\frac{\partial w_2}{\partial v} + m w_1 + m_u w_3 + b' w_4 &= q,\\
\frac{\partial w_3}{\partial v} + c' w_4 &= 0,\\
\frac{\partial w_4}{\partial v} + w_2 + d' w_4 &= 0.
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

And x must satisfy the system of the equations

$$\frac{w_3}{p}\frac{\partial^2 x}{\partial u^2} + \frac{w_4}{q}\frac{\partial^2 x}{\partial v^2} + \left(\frac{w_1}{p} - \frac{w_3 p_u}{p^2}\right)\frac{\partial x}{\partial u} + \left(\frac{w_2}{q} - \frac{w_4 q_u}{q^2}\right)\frac{\partial x}{\partial v} = x,$$

$$\frac{\partial^2 x}{\partial u \partial v} = \frac{p_v}{p}\frac{\partial x}{\partial u} + \frac{q_u}{q}\frac{\partial x}{\partial v}.$$

The asymptotic lines S_y, S_z and S are given by the equations

$$\frac{d}{m} du^2 + dv^2 = 0,$$

$$du^2 + \frac{c'}{n} dv^2 = 0,$$

$$\frac{pw_4}{qw_3}du^2-dv^2=0.$$

If the given congruence is a W-congruence and the asymptotic lines of the focal surfaces and S-correspond, we must have

$$\frac{d}{m}=\frac{n}{c'}=\frac{-pw_4}{qw_3}, \quad \text{(16)}$$

or

$$\frac{dw_3}{w_4}=-\frac{mp}{q},$$

$$\frac{c'w_4}{w_3}=-\frac{nq}{p}.$$

These are equivalent to

$$\frac{1}{w_4}\frac{\partial w_4}{\partial u}=\frac{1}{q}\frac{\partial q}{\partial u},$$

$$\frac{1}{w_1}\frac{\partial w_3}{\partial v}=\frac{1}{p}\frac{\partial p}{\partial v}.$$

Integrating, we have

$$w_3=p\,U(u),$$

$$w_4=q\,U(v).$$

where $U(u)$ and $V(v)$ are certain functions of u and v only respectively.

Substituting the above values of w_3, w_4 in (16) we have

$$\frac{d}{m}=\frac{n}{c'}=-\frac{V(v)}{U(u)}.$$

Therefore by the transformation

$$\bar{u}=\int\frac{du}{\sqrt{U(u)}},$$

$$\bar{v}=\int\frac{dv}{\sqrt{V(v)}}.$$

we can reduce c, d, m and n so as to the satisfy the relations

$$d = -m,$$
$$c' = -n.$$

Let us suppose that we have performed this transformation, Then we have

$$\frac{pw_4}{qw_3} = 1,$$

from which we can easily see that

$$\left.\begin{aligned} kw_3 &= p, \\ kw_4 &= q, \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (17)$$

where k is a constant.

From these relations and (15) we have

$$\left.\begin{aligned} kw_1 &= -(p_u + cp), \\ kw_2 &= -(q_v + d'q). \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (18)$$

By (14), (15), (17) and (18), p and q must satisfy the syetem of equations

$$\left.\begin{aligned} &\frac{\partial p}{\partial v} = nq, \qquad \frac{\partial q}{\partial u} = mp, \\ &\frac{\partial^2 p}{\partial u^2} = (a - c_u - k)p + (n_v - nd')q - c\frac{\partial p}{\partial u} - n\frac{\partial q}{\partial v}, \\ &\frac{\partial^2 q}{\partial v^2} = (m_u - mc)p + (b' - d'_u - k)q - m\frac{\partial p}{\partial u} - d'\frac{\partial q}{\partial v}. \end{aligned}\right\} (19)$$

When

$$d = -m, \qquad c' = -n.$$

The conditions (10) become

$$\left.\begin{aligned} &c: \quad \text{a function of } u \text{ only}, \\ &d': \quad \text{a function of } v \text{ only}, \\ &b = m_v + md', \qquad a' = n_u + nc, \\ &m_{uu} - m_{vv} - md'_v - m_v d' - m_u c = m(a - b'), \\ &n_{vv} - n_{uu} - mc'_u - n_u c - u_v d' = n(a - b'), \\ &2\frac{\partial}{\partial u}(mn) = \frac{\partial a}{\partial v}, \\ &2\frac{\partial}{\partial v}(mn) = \frac{\partial b'}{\partial n}. \end{aligned}\right\} (10')$$

We can easily see that the system (19) is completely integrable by (10′). Therefore, we have four linearly independent pairs of solutions for (19).

The surface (S) which is in the relation above described with the given congruence, is given by the system

$$\left.\begin{aligned} \frac{\partial^2 x}{\partial u^2}+\frac{\partial^2 x}{\partial v^2}&=\left(c+\frac{2p_u}{p}\right)\frac{\partial x}{\partial u}+\left(d'+\frac{2q_v}{q}\right)\frac{\partial x}{dv}+kx,\\ \frac{\partial^2 x}{\partial u\partial v}&=\frac{p_v}{p}\frac{\partial x}{\partial u}+\frac{q_u}{q}\frac{\partial x}{\partial v}. \end{aligned}\right\} \quad (20)$$

Therefore, we know that *if a surface* (S) *has W-congruence* (D) *as its congruence* Γ, *and the asymptotic lines of* (S) *correspond to those of the focal surfaces of* (D), *the surface* (S) *and the focal surfaces admit the projeceive deformation and the edges of regression of the developables of* (D) *form the conjugate net of projective deformation on the focal surface on which they lie with their conjugate lines.*

Reciprocally, if W-congruence (D) is such that the focal surfaces of it admit projective deformation and edges of regression of the developables of it are the lines of the conjugate net of projective deformation of the focal surface on which they lie, there are surfaces which have the congruence (D) as a congruence Γ of them and of which the asymptotic lines correspond to those of the focal surfaces of (D).

We have elsewhere[1] proved that if there is an isothermal conjugate net on a surface (S) such that the asymptotic lines of (S) and its minus-first or first Laplace-Dardoux transform correspond, (S) admits the projective deformation and the conjugate net is that of projective deformation,

Therefore, we may say as follows:—

If the edges of regression of the developables of a W-congruence form an isothermal conjugate net with their conjugate lines on the focal surface on which they lie, there are surfaces which have that congruence as a congruence Γ *of them and of which the asymptotic lines correspond to those of the focal surfaces of that congruence.*

Consider a congruence D_λ which is given by the system

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[1] J. Kanitani, Mem. Coll. Sci., **6**, 29–43 (1922).

$$\left.\begin{aligned}
&\frac{\partial y}{\partial v}=mz, \qquad \frac{\partial z}{\partial u}=ny,\\
&\frac{\partial^2 y}{\partial u^2}=(a+\lambda)y+bz+c\frac{\partial y}{\partial u}-m\frac{\partial z}{\partial v},\\
&\frac{\partial^2 y}{\partial v^2}=a'y+(b'+\lambda)z-n'\frac{\partial y}{\partial u}+d'\frac{\partial z}{\partial v},\\
&\qquad(\lambda:\ \text{const.})
\end{aligned}\right\}\quad(21)$$

which is completely integrable by (10′).

Denote by D_0 the congruence given by the system which is obtained by putting $\lambda=0$ in (21) and by $S_{y\lambda}$, $S_{z\lambda}$ the focal surfaces which are described by the points (y), (z) respectively, which are defined by (21).

Evidentiy $S_{y\lambda}$, and $S_{z\lambda}$ are projectively deformable to S_y and S_z respectivly.

The surface S_λ which is given by

$$\frac{\partial^2 x}{\partial u^2}+\frac{\partial^2 x}{\partial v^2}=\left(c+\frac{2p_u}{p}\right)\frac{\partial x}{\partial n}+\left(d'+\frac{2q_v}{q}\right)\frac{\partial x}{\partial v}+(k+\lambda)x,$$

$$\frac{\partial^2 x}{\partial u\partial v}=\frac{p_v}{p}\frac{\partial x}{\partial u}+\frac{q_u}{q}\frac{\partial x}{\partial v}.$$

where p, q are the same function as those which appear in (20), *i.e.*, a pair of solutions of (19), has the same relation with D_λ as those which S has with D_0 and is projectively deformable to S.

Therefore we have the following theorem.

Let D be a congruence which is formed by tangent lines of a family of curves of a conjugate net of projective deformation of a surface S of the first kind which affords only ∞^1 surfaces which are projectively deformable to it, and Σ be a surface which has D as its congruence Γ and of which the asymptotic lines correspond to those of the focal surfaces of D, that is, those of S.

If a surface S_λ be projectively deformable to S, there are surfaces projectively deformable to Σ among those which have the same relations with S_λ as Σ has with S.

§ 3.

In this paragraph we shall consider a surface (S) which are in the following relation with the congruence which is given by system (9).

1° There is a one to one correspondence between the points on (S) and the lines of the given congruence.

2° A point on (S) lies on the corresponding line of the congruence.

3° Developables of the congruence cut (S) in a conjugate net.

Coordinates (x) of such surface are given by[1]

$$mx = p\frac{\partial y}{\partial v} - \frac{\partial p}{\partial v}y,$$

where p is a solution of the equation

$$\frac{\partial^2 p}{\partial u \partial v} = \frac{m_u}{m}p + mnp,$$

which is satisfied by the coordinates (y) of a point on the focal surface S_y.

If we put

$$q = \frac{1}{m}\frac{\partial p}{\partial v},$$

we have

$$x = pz - qy, \qquad \text{..... (22)}$$

$$\left.\begin{aligned} \frac{\partial p}{\partial v} &= mq, \\ \frac{\partial q}{\partial u} &= np. \end{aligned}\right\} \qquad \text{..... (23)}$$

From (22) and (23) we have

$$\frac{\partial x}{\partial u} = \frac{\partial p}{\partial u}z - q\frac{\partial y}{\partial u},$$

$$\frac{\partial x}{\partial v} = p\frac{\partial z}{\partial v} - \frac{\partial q}{\partial u}y,$$

from which follows

$$\frac{\partial^2 x}{\partial u \partial v} = \frac{q_v}{q}\frac{\partial x}{\partial u} + \frac{p_u}{p}\frac{\partial x}{\partial v} + \left(mn - \frac{p_u q_v}{pq}\right)x.$$

If we put

$$w = ap + bq + c\frac{\partial p}{\partial u} + d\frac{\partial q}{\partial v} - \frac{\partial^2 p}{\partial u^2},$$

---

[1] Darboux, II., chap. X.

$$\tau = a'p + b'q + c'\frac{\partial p}{\partial u} + d'\frac{\partial q}{\partial v} - \frac{\partial^2 q}{\partial^2 v},$$

we have from (10), (22) and (23).

$$\frac{\partial^2 x}{\partial u^2} = \left(\frac{np}{q} + c\right)\frac{\partial x}{\partial u} - \frac{dq}{p}\frac{\partial x}{\partial v} + \left(a + \frac{dq_v}{p} - \frac{np_u}{q}\right)x - wz,$$

$$\frac{\partial^2 x}{\partial v^2} = -\frac{c'p}{q}\frac{\partial x}{\partial u} + \left(d' + \frac{mq}{p}\right)\frac{\partial x}{\partial v} + \left(b' + \frac{c'p_u}{q} - \frac{mq_v}{p}\right)x + \tau y.$$

from which we know that x satisfies the equation of the form

$$\frac{\partial^2 x}{\partial u^2} = A\frac{\partial^2 x}{\partial v^2} + B\frac{\partial x}{\partial u} + C\frac{\partial x}{\partial v} + Dx,$$

where

$$A = -\frac{wq}{\tau p}.$$

If the asymptotic lines of (S) and the focal surfaces S_y, S_z correspond, we must have

$$\frac{d}{m} = \frac{n}{c'} = \frac{-\tau q}{\tau p}. \qquad \text{..... (24)}$$

But we know from (10) that w and τ satisfy the equation

$$\left.\begin{aligned}\frac{\partial w}{\partial v} &= -d\tau,\\ \frac{\partial w}{\partial u} &= -c'w.\end{aligned}\right\} \qquad \text{..... (25)}$$

From (24) and (25) we can see, as in the preceding paragraph, that by the transformation of the form

$$\bar{u} = U(u), \qquad \bar{v} = V(v),$$

m, n, c, c may be reduced so as to satisfy the equation

$$d = -m,$$
$$c' = -n.$$

Let us suppose that we have performed the above transformation. Then we have

$$w = kp,$$
$$\tau = kq,$$

where k is a constant, and p and q satisfy the system of the equations

$$\frac{\partial p}{\partial v} = mq, \qquad \frac{\partial q}{\partial u} = np,$$

$$\frac{\partial^2 p}{\partial u^2} = (a-b)p + bq + c\frac{\partial p}{\partial u} + d\frac{\partial q}{\partial v},$$

$$\frac{\partial^2 q}{\partial v^2} = a'p + (b'-k)p + c'\frac{\partial p}{\partial u} + d'\frac{\partial q}{\partial v},$$

which is completely integrable by (10′).

The surface (S) which satisfies our conditions is given by the system

$$\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2} = \left(c + \frac{2q_u}{q}\right)\frac{\partial x}{\partial u} + \left(d' + \frac{2p_v}{p}\right)\frac{\partial \lambda}{\partial v} + \left(a + b' - \frac{2mq_v}{p} - \frac{2np_u}{q} - k\right)x.$$

$$\frac{\partial^2 x}{\partial u \partial v} = \frac{q_v}{q}\frac{\partial x}{\partial u} + \frac{p_u}{p}\frac{\partial x}{\partial v} + \left(mn - \frac{p_u q_v}{pq}\right)x.$$

Therefore we know that the theorems in § 2 also hold, if we substitute congruence I' for the congruence I.

§ 4.

Consider two surfaces S_x and S_X which are in the relations explained in § 3 with the given congruence given by the system.

$$\frac{\partial y}{\partial v} = mz, \qquad \frac{\partial z}{\partial u} = ny,$$

$$\frac{\partial^2 x}{\partial u^2} = ay + bz + c\frac{\partial y}{\partial u} - m\frac{\partial z}{\partial u},$$

$$\frac{\partial^2 x}{\partial v^2} = a'y + b'z - n\frac{\partial y}{\partial u} + d'\frac{\partial z}{\partial v},$$

of which the conditions of completely integrability are (10′).

Suppose S_x and S_X are described by the points (x) and (X) respectively such that.

$$\left.\begin{aligned} x &= pz - qy, \\ X &= Pz - Qy, \end{aligned}\right\} \quad \cdots\cdots (26)$$

where (p, q) and (P, Q) are pairs of the solutions of the system

$$\left.\begin{aligned} &\frac{\partial p}{\partial v} = mq, \qquad \frac{\partial q}{\partial u} = np, \\ &\frac{\partial^2 p}{\partial u^2} = (a-k)p + bq + c\frac{\partial p}{\partial u} - m\frac{\partial q}{\partial v}, \\ &\frac{\partial^2 q}{\partial v^2} = a'p + (b'-k)q - n\frac{\partial p}{\partial u} + d'\frac{\partial q}{\partial v}, \end{aligned}\right\} \quad (27)$$

and the system

$$\left.\begin{aligned} &\frac{\partial P}{\partial v} = mQ, \qquad \frac{\partial Q}{\partial u} = nP, \\ &\frac{\partial^2 P}{\partial u^2} = (a-k_1)P + bQ + c\frac{\partial P}{\partial u} - m\frac{\partial Q}{\partial v}, \\ &\frac{\partial^2 Q}{\partial v^2} = a'P + (b'-K)Q - n\frac{\partial P}{\partial u} + d'\frac{\partial Q}{\partial v}, \end{aligned}\right\} \quad (28)$$

respectively.

Then x satisfies the system of equations

$$\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2} = \left(c + \frac{2q_u}{q}\right)\frac{\partial x}{\partial u} + \left(d' + \frac{2p_v}{p}\right)\frac{\partial x}{\partial v} + \left(a + b' - \frac{2mq_v}{p} - \frac{2np_v}{q} - k\right)x,$$

$$\frac{\partial^2 x}{\partial u \partial v} = \frac{q_v}{q}\frac{\partial x}{\partial u} + \frac{p_u}{p}\frac{\partial x}{\partial v} + \left(mn - \frac{p_u q_v}{pq}\right)x,$$

while X satisfies the system of the equations

$$\frac{\partial^2 X}{\partial u^2} + \frac{\partial^2 X}{\partial v^2} = \left(c + \frac{2q_u}{q}\right)\frac{\partial X}{\partial u} + \left(d' + \frac{2p_v}{p}\right)\frac{\partial X}{\partial v} + \left(a + b' - \frac{2mq_v}{p} - \frac{2np_v}{q} - k_1\right)X,$$

$$\frac{\partial^2 X}{\partial u \partial v} = \frac{q_v}{q}\frac{\partial X}{\partial u} + \frac{p_u}{p}\frac{\partial X}{\partial v} + \left(mn - \frac{p_u q_v}{pq}\right)X.$$

Put

$$\theta = pQ - qP, \qquad \cdots\cdots (29)$$

Then we have from (27) and (28).

$$\frac{\partial\theta}{\partial u} = \frac{\partial p}{\partial u} Q - q \frac{\partial P}{\partial u},$$

$$\frac{\partial\theta}{\partial v} = p \frac{\partial Q}{\partial v} - \frac{\partial q}{\partial v} Q,$$

$$\frac{\partial^2\theta}{\partial u \partial v} = \frac{\partial p}{\partial u}\frac{\partial Q}{\partial v} - \frac{\partial P}{\partial u}\frac{\partial q}{\partial v} + mn\theta,$$

$$\frac{\partial^2\theta}{\partial u^2} = \left(c + \frac{q_u}{q}\right)\frac{\partial\theta}{\partial u} + \frac{p_v}{p}\frac{\partial\theta}{\partial v} + \left(a - \frac{mq_v}{p} - \frac{np_u}{q}\right)\theta - kpQ + k_1 qP,$$

$$\frac{\partial^2\theta}{\partial v^2} = \frac{q_u}{q}\frac{\partial\theta}{\partial u} + \left(d' + \frac{p_v}{p}\right)\frac{\partial\theta}{\partial v} + \left(b' - \frac{np_u}{q} - \frac{mq_v}{p}\right)\theta - k_1 pQ + kqP,$$

from which follow

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = \left(c + \frac{2q_u}{q}\right)\frac{\partial\theta}{\partial u} + \left(d' + \frac{p_v}{p}\right)\frac{\partial\theta}{\partial v} + \left\{a + b' - \frac{2mq_v}{p} - \frac{2np_u}{q} - (k + k_1)\right\}\theta,$$

$$\frac{\partial^2\theta}{\partial u \partial v} = \frac{q_v}{q}\frac{\partial\theta}{\partial u} + \frac{p_u}{p}\frac{\partial\theta}{\partial v} + \left(mn - \frac{p_u q_v}{pq}\right)\theta,$$

Similarly we have

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = \left(c + \frac{2Q_u}{Q}\right)\frac{\partial\theta}{\partial u} + \left(d' + \frac{P_v}{P}\right)\frac{\partial\theta}{\partial v} + \left\{a + b' - \frac{2mQ_v}{P} - \frac{2nP_u}{Q} - (k + k_1)\right\}\theta,$$

$$\frac{\partial^2\theta}{\partial u \partial v} = \frac{P_v}{P}\frac{\partial\theta}{\partial u} + \frac{P_u}{P}\frac{\partial\theta}{\partial v} + \left(mn - \frac{P_u Q_v}{PQ}\right)\theta.$$

But from (26) and (29), we easily see that

$$Q\left(\theta\frac{\partial x}{\partial u}-\frac{\partial\theta}{\partial u}x\right)\equiv q\left(\theta\frac{\partial X}{\partial u}-\frac{\partial\theta}{\partial u}X\right),$$

$$P\left(\theta\frac{\partial x}{\partial v}-\frac{\partial\theta}{\partial v}x\right)\equiv p\left(\frac{\partial X}{\partial v}-\frac{\partial\theta}{\partial v}X\right).$$

Therefore we have the following theorem:—

Let D be a W-congruence such that the edges of regression of its developable surfaces from an isothermal conjugate net on the focal surface on which they lie with their conjugate lines, and S and T be two surfaces which have D as their congruence Γ' and of which the asymptotic lines correspond to those of the focal surfaces of D. Then the lines of intersection of the tangent planes of S and T at the corresponding points from a congruence Γ conjugate to S, as well as to T such that the asymptotic lines of S, T and their focal surfaces correspond.

Let (P_1, Q_1), (P_2, Q_2), (P_3, Q_4) and (P_4, Q_Q) be four lieearly indedendent pairs of solution of the system (28).

Then θ_1, θ_2, θ_3, θ_4 such that

$$\theta_i=pQ_i-qP_i \quad (i=1,\ 2,\ 3,\ 4).$$

are also linearly independent.

For if there be a relation such that

$$c_1\theta_1+c_2\theta_2+c_3\theta_3+c_4\theta_4=0,$$

we have by (27) and (28),

$$\sum_{i=1}^{4}\frac{\partial}{\partial v}\left(\frac{c_iP_i}{p}\right)=0,$$

$$\sum_{i=1}^{4}\frac{\partial}{\partial u}\left(\frac{c_iQ_i}{q}\right)=0,$$

Integrating, we have

$$\sum_{i=1}^{4}c_iP_i=U(u)p,$$

$$\sum_{i=1}^{4}c_iQ_i=V(v)q.$$

Differentiating the former by v and comparing the result with the

latter, we have

$$U(u)=V(v)=C,$$

where C is a constant.

If $k_1 \neq k$, we must have $C=0$.

This contradicts our assumption.

When $k_1=k$, we may take

$$P_1=p, \qquad Q_1=q,$$

and reduce to a similar contradiction.

Therefore, we know that *congruences Γ conjugate to S such that the asymptotic lines of their focal surfaces correspond to those of S, are those which are formed by intersections of tangent planes at corresponding points of S and a surface which has the same relation as S with D.*

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